THE WEAK ASYMPTOTICS METHOD. INTERACTION OF SHOCKS IN GASES

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Abstract

The problem of uniqueness for shock waves interaction in isothermal gas dynamics is under consideration. Using the weak asymptotics method, an uniform in time asymptotic solution has been constructed for two interacting shocks.

1 Prehistory

The relations for parameters of single shock waves (the so-called Rankine-Hugoniot conditions) have been found by Rankine and Hugoniot. The problem of arbitrary shocks decay in gases has been solved by Riemann. This implied the description of shock waves interaction. Therefore, this problem seemed to be solved and closed.

However, when mathematicians started to define shock wave solutions in the sense of distributions, a problem of nonuniqueness of weak solutions appeared. It is well-known that for single shocks this problem has been overcome by O. Oleinik and S. Kruzhkov for scalar equations and by P. Lax for hyperbolic systems (the so-called entropy conditions have been found). At the same time, similar problem for interacting shocks remained unsolved.

The unique result here has been obtained by G. Whitham in 50-th of the last century for the Hopf equation (inviscid Burgers equation) with quadratic nonlinearity

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} = 0, \tag{1}$$

but the techniques used does not allow to consider more general problems including the Hopf equation with convex nonlinearity

$$\frac{\partial u}{\partial t} + \frac{\partial \Phi(u)}{\partial x} = 0, \qquad \Phi'' > 0. \tag{2}$$

The main obstacle here consists of the following. In order to prove the uniqueness we follow the method of vanishing viscosity by Oleinik and pass from the hyperbolic equation (2) (or (1)) to a regularization, in the simplest case to the Burgers equation

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial \Phi(u_\varepsilon)}{\partial x} = \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2}, \qquad \varepsilon << 1. \tag{3}$$

Instead of nonsmooth initial data for the Hopf equation

$$u|_{t=0} = A_1 H(x) + A_2 H(x - x^0), \tag{4}$$

where $H(x)$ is the Heaviside function, $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$, $A_i = \text{const}$, we consider smooth initial data

$$u_\varepsilon|_{t=0} = A_1 \omega\big(\frac{x}{\varepsilon}\big) + A_2 \omega\big(\frac{x - x^0}{\varepsilon}\big), \tag{5}$$

where $\omega(x/\varepsilon)$ is a regularization of the Heaviside function. Obviously, for any fixed $\varepsilon = \text{const}$ the solution of the problem (3), (5) exists and it is smooth and unique. However, to find the limiting solution $u = \lim_{\varepsilon\to 0} u_\varepsilon$ for a time after the interaction we have to solve the problem (3), (5) exactly or, at least to describe the solution in detail. Unfortunately, the exact solution can be found only in the case of quadratic nonlinearity $\Phi(u) = u^2$ (using the Hopf-Cole substitution). Moreover, any traditional asymptotic method does not work here.

2 The weak asymptotics method

The progress in the problem of nonlinear waves interaction has been achieved only recently in the framework of the weak asymptotics method (Danilov, Omel'yanov, Shelkovich, [1] - [3]). The main advantage of this approach is the reduction of the problem of describing of nonlinear waves interaction to qualitative analysis of some system of ordinary differential equations (instead of analysis of partial differential equations).

This method takes into account the fact that solutions of regularized problems, which are smooth for $\varepsilon > 0$, become non-smooth in the limit as $\varepsilon \to 0$. So we will treat these solutions as mappings $C^\infty(0, T; C^\infty(\mathbb{R}^1_x))$ for $\varepsilon > 0$ and only as $C(0, T; \mathcal{D}'(\mathbb{R}^1_x))$ uniformly in $\varepsilon \geq 0$. Next, since it impossible to find exact solutions, we will construct asymptotic solutions treating the smallness of remainders in the weak sense. For example, u_ε is a weak asymptotic solution of the equation (3) with precision $O_{\prime}(\varepsilon)$ if for any $\psi \in (^1_x)$ the following relation holds

$$\frac{d}{dt}\int_{-\infty}^{\infty} u_\varepsilon \psi dx - \int_{-\infty}^{\infty} \Big\{\Phi(u_\varepsilon)\frac{\partial \psi}{\partial x} + \varepsilon u_\varepsilon \frac{\partial^2 \psi}{\partial x^2}\Big\} dx = O(\varepsilon). \tag{6}$$

Another important remark is that solutions under consideration correspond in the limit to elements of some subalgebras with shifts in the space of generalized functions. For example, let $\omega(\frac{x}{\varepsilon})$ be a regularization of the Heaviside function $H(x)$,

$$\omega(\frac{x}{\varepsilon}) = H(x) + O_{\mathcal{D}'}(\varepsilon) \to H(x) \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad \mathcal{D}' \quad \text{sense.}$$

Then

$$\omega^2(\frac{x}{\varepsilon}) = H(x) + O_{\mathcal{D}'}(\varepsilon) \to H(x) \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad \mathcal{D}' \quad \text{sense.}$$

Moreover, for the product of $H(x)$ and $H(x - \varphi)$ we can write only the nonuniform in φ formula

$$H(x)H(x-\varphi) = H(x-\varphi) \quad \text{if} \quad \varphi > 0, \qquad H(x)H(x-\varphi) = H(x) \quad \text{if} \quad \varphi < 0.$$

However, passing to generalized functions we obtain the uniform formula

$$\omega\big(\frac{x}{\varepsilon}\big)\omega\big(\frac{x-\varphi}{\varepsilon}\big)=H(x)B\big(\frac{\varphi}{\varepsilon}\big)+H(x-\varphi)\Big(1-B\big(\frac{\varphi}{\varepsilon}\big)\Big)+O_{\mathcal{D}'}(\varepsilon),$$

where $B(z)\in C^\infty$ is such that $B(z)\to 1$ as $z\to-\infty$ and $B(z)\to 0$ as $z\to+\infty$. Realization of these ideas allowed us to solve some nontrivial problems. In particular, the interaction of solitons for nonintegrable versions of the KdV and sine-Gordon equations have been described and the uniqueness of shock waves interaction for the Hopf equation with convex nonlinearities (2), (4) has been proved, [1] - [3].

3 Shock waves interaction in gas dynamics

Consider the simplest isothermal version of the gas dynamics equations

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0,\qquad \rho\Big(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\Big)+\frac{\partial p}{\partial x}=0,\qquad p=c_0^2\rho. \tag{7}$$

Let us restrict ourselves by the case of two shocks with opposite directions of motion

$$\rho|_{t=0}=\rho_0+e_1H(-x+x_1^0)+e_2H(x-x_2^0),$$
$$u|_{t=0}=u_1H(-x+x_1^0)+u_2H(x-x_2^0), \tag{8}$$

where $e_1=\rho_1-\rho_0$, $e_2=\rho_2-\rho_0$ and we assume that $\rho_i>\rho_0>0$, $i=1,2$, $u_1>0>u_2$ and $x_1^0<x_2^0$. The existence of the problem (7),(8) solution is well-known. Our aim is the proof of the uniqueness of the solution.

Before the time of interaction the solution is the sum of isolated shocks,

$$\rho=\rho_0+e_1H(-x+\varphi_{10}(t))+e_2H(x-\varphi_{20}(t)),$$
$$u=u_1H(-x+\varphi_{10}(t))+u_2H(x-\varphi_{20}(t)), \tag{9}$$

where

$$\varphi_{10}(t)=c_0\sqrt{\rho_1/\rho_0}\,t+x_1^0,\qquad \varphi_{20}(t)=-c_0\sqrt{\rho_2/\rho_0}\,t+x_2^0.$$

In order to obtain a uniform in time description of the process of interaction, let us pass to a regularized problem

$$\frac{\partial\rho_\varepsilon}{\partial t}+\frac{\partial}{\partial x}(\rho_\varepsilon u_\varepsilon)=0,$$
$$\frac{\partial}{\partial t}(\rho_\varepsilon u_\varepsilon)+\frac{\partial}{\partial x}\big(\rho_\varepsilon u_\varepsilon^2+p_\varepsilon\big)=\varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2},\quad p_\varepsilon=c_0^2\rho_\varepsilon, \tag{10}$$
$$\rho_\varepsilon|_{t=0}=\rho_0+e_1\omega\big(\frac{-x+x_1^0}{\varepsilon}\big)+e_2\omega\big(\frac{x-x_2^0}{\varepsilon}\big),$$
$$u_\varepsilon|_{t=0}=u_1\omega\big(\frac{-x+x_1^0}{\varepsilon}\big)+u_2\omega\big(\frac{x-x_2^0}{\varepsilon}\big),$$

where $\omega(x/\varepsilon)$ is a regularization of the Heaviside function $H(x)$.

Of course, it is impossible to find the exact solution of the problem (10). Therefore we will construct a weak asymptotic $\mod O_{\mathcal{D}'}(\varepsilon)$ solution. This means that the following relations

$$\frac{d}{dt}\int_{-\infty}^{\infty}\rho_\varepsilon\psi_1 dx-\int_{-\infty}^{\infty}\rho_\varepsilon u_\varepsilon\frac{\partial\psi_1}{\partial x}dx=O(\varepsilon),$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}\rho_\varepsilon u_\varepsilon\psi_2 dx-\int_{-\infty}^{\infty}\Big((\rho_\varepsilon u_\varepsilon^2+p_\varepsilon)\frac{\partial\psi_1}{\partial x}+\varepsilon u_\varepsilon\frac{\partial^2\psi_2}{\partial x^2}\Big)dx=O(\varepsilon), \tag{11}$$

hold for any test functions $\psi_i\in\mathcal{D}(\mathbb{R}^1)$, $i=1,2$.

Let us write the anzatz in the following form

$$\rho_\varepsilon=\rho_0+e_1\omega\big(\frac{-x+\Phi_1}{\varepsilon}\big)+e_2\omega\big(\frac{x-\Phi_2}{\varepsilon}\big)+R\omega\big(\frac{-x+\Phi_1}{\varepsilon}\big)\omega\big(\frac{x-\Phi_2}{\varepsilon}\big),$$

$$u_\varepsilon=u_1\omega\big(\frac{-x+\Phi_1}{\varepsilon}\big)+u_2\omega\big(\frac{x-\Phi_2}{\varepsilon}\big)+U\omega\big(\frac{-x+\Phi_1}{\varepsilon}\big)\omega\big(\frac{x-\Phi_2}{\varepsilon}\big), \tag{12}$$

where

$$\Phi_i=\varphi_{i0}(t)+\psi_0(t)\varphi_{i1}(\tau),\quad \psi_0(t)=\varphi_{20}(t)-\varphi_{10}(t),\quad \tau=\frac{\psi_0(t)}{\varepsilon},$$

functions $\varphi_{i1}(\tau)\in C^\infty$, $R=R(\tau)\in C^\infty$, and $U=U(\tau)\in C^\infty$ are such that

$$\varphi_{i1}(\tau)\to 0\quad\text{as}\quad\tau\to+\infty,\quad \varphi_{i1}(\tau)\to\bar{\varphi}_{i1}=\text{const}\quad\text{as}\quad\tau\to-\infty,$$

$$R\quad\text{and}\quad U\quad\text{are slowly increasing functions}\quad\text{as}\quad\tau\to+\infty,$$

$$R(\tau)\to\bar{R}=\text{const},\quad U(\tau)\to\bar{U}=\text{const}\quad\text{as}\quad\tau\to-\infty.$$

It is easy to calculate the weak asymptotic expansions for the functions (11)

$$\rho_\varepsilon=\rho_0+\{e_1+RB\}H(-x+\Phi_1)+\{e_2-RB\}H(x-\Phi_2)+O_{\mathcal{D}'}(\varepsilon),$$

$$u_\varepsilon=\{u_1+UB\}H(-x+\Phi_1)+\{u_2-UB\}H(x-\Phi_2)+O_{\mathcal{D}'}(\varepsilon), \tag{13}$$

where the function $B(\tau)\in C^\infty$ has the properties

$$B(\tau)\to 0\quad\text{as}\quad\tau\to+\infty\qquad B(\tau)\to 1\quad\text{as}\quad\tau\to-\infty.$$

The next step is the calculations of the products $\rho_\varepsilon u_\varepsilon$ and $\rho_\varepsilon u_\varepsilon^2$. In fact, the weak asymptotic expansions of them have the same structure as presented in (13). For example,

$$\rho_\varepsilon u_\varepsilon=\{\rho_1u_1+G_1(\tau)\}H(-x+\Phi_1)+\{\rho_2u_2-G_1(\tau)\}H(x-\Phi_2)+O_{\mathcal{D}'}(\varepsilon),$$

where G_1 is a function written in terms of R, U, and some convolutions. So, substituting the anzatz (12) into the relations (11) we obtain the following equalities

$$D_{i1}(\tau)\delta(x-\Phi_1)+D_{i2}(\tau)\delta(x-\Phi_2)=0,\qquad i=1,2,$$

where

$$D_{i1} = \frac{d\varphi_{10}}{dt} + \psi_{0t}\frac{d}{d\tau}(\tau\varphi_{11}) + K_{i1}\Big(\frac{d}{d\tau}(RB),\ \frac{d}{d\tau}(UB), RB, UB, \tau\Big),$$

$$D_{i2} = \frac{d\varphi_{20}}{dt} + \psi_{0t}\frac{d}{d\tau}(\tau\varphi_{21}) + K_{i2}\Big(\frac{d}{d\tau}(RB),\ \frac{d}{d\tau}(UB), RB, UB, \tau\Big).$$

Obviously, this implies the equations

$$D_{ij} = 0, \qquad i, j = 1, 2. \tag{14}$$

The equations (14) are, perhaps, the most important result of the construction since they can be treated as a generalization of the Rankine-Hugoniot conditions for the case of two interacting shocks. Indeed, consider a time before the interaction. Since $\tau = (\varphi_{20}(t) - \varphi_{10}(t))/\varepsilon$ and $\varepsilon \to 0$, this time t corresponds to the limit as $\tau \to +\infty$. It is easy to check that equations (14) transform in this limit to the following form

$$e_k \frac{d\varphi_{k0}}{dt} = \rho_k u_k, \quad \rho_k u_k \frac{d\varphi_{k0}}{dt} = \rho_k u_k^2 + c_0^2 e_k, \quad k = 1, 2. \tag{15}$$

Obviously, we obtain the Rankine-Hugoniot conditions for the original shocks before the interaction. Let us consider the limit $\tau \to -\infty$ which corresponds to time t after the interaction. Let us denote

$$\Phi_k \to \bar{\Phi}_k(t), \quad \rho_0 + e_1 + e_2 + RB \to \rho^*, \quad u_1 + u_2 + UB \to u^*.$$

Then we obtain from (14)

$$(\rho^* - \rho_2)\frac{d\bar{\Phi}_1}{dt} = \rho^* u^* - \rho_2 u_2, \quad (\rho^* u^* - \rho_2 u_2)\frac{d\bar{\Phi}_1}{dt} = \rho^* u^{*2} - \rho_2 u_2^2 + c_0^2(\rho^* - \rho_2), \tag{16}$$

$$(\rho^* - \rho_1)\frac{d\bar{\Phi}_2}{dt} = \rho^* u^* - \rho_1 u_1, \quad (\rho^* u^* - \rho_1 u_1)\frac{d\bar{\Phi}_2}{dt} = \rho^* u^{*2} - \rho_1 u_1^2 + c_0^2(\rho^* - \rho_1). \tag{17}$$

We came again to the Rankine-Hugoniot conditions for the pair of shocks $\{(\rho^*, u^*),(\rho_2, u_2)\}$ and $\{(\rho^*, u^*),(\rho_1, u_1)\}$ which propagate over the background (ρ_2, u_2) and (ρ_1, u_1), respectively. Moreover,

$$\rho^* = \frac{\rho_1\rho_2}{\rho_0}, \qquad u^* = u_1 + u_2. \tag{18}$$

This is just the answer which has been found by Riemann.

It is very important to note that the limiting relations (15) and (16),(17) do not depend on the way of regularization of the original problem.

The last step is the proof of existence of admissible functions φ_{i1}, R, and U. To this aim we pass from the equations (14) to the dynamical system

$$\frac{d\widetilde{R}}{d\tau} = \mathcal{F}_1(\widetilde{R}, \widetilde{U}, \sigma), \quad \frac{d\widetilde{U}}{d\tau} = \mathcal{F}_2(\widetilde{R}, \widetilde{U}, \sigma), \quad \frac{d\sigma}{d\tau} = \mathcal{F}_3(\widetilde{R}, \widetilde{U}, \sigma) \tag{19}$$

for the functions $\widetilde{R} = RB$, $\widetilde{U} = UB$, $\sigma = (\Phi_2 - \Phi_1)/\varepsilon$. We set a scattering type conditions

$$\widetilde{R} \to 0, \quad \widetilde{U} \to 0, \quad \frac{\sigma}{\tau} \to 1 \quad \text{as} \quad \tau \to +\infty \tag{20}$$

and have to learn the solution as $\tau \to -\infty$.

To understand the structure of the phase portrait it is necessary to take into account that there are surfaces of singularity $\Gamma_i \subset \mathbb{R}^3$ where $\mathcal{F}_k \to \infty$ and curves of equilibrium $\gamma_j \subset \mathbb{R}^3$ where $\mathcal{F}_1 = 0$ and $\mathcal{F}_2 = 0$. Let, for simplicity, $\rho_1 = \rho_2$. Then the most important curve γ_1 lies on the plane $\widetilde{U} = 0$ and has the limiting points $a^{\pm\infty}$ as $\tau \to \pm\infty$. The point $a^{+\infty}$ is the limit of γ_1 as $\tau \to +\infty$ and it easy to check that the "initial data" (20) hold there. The point $a^{-\infty}$ is the limit of γ_1 as $\tau \to -\infty$, and there

$$\sigma \to \frac{\rho_0}{\rho_1}\tau, \quad \widetilde{R} \to \frac{(\rho_1 - \rho_0)^2}{\rho_0}, \quad \widetilde{U} = 0.$$

Moreover, $a^{-\infty}$ is a point similar to the stable node but with polinomial behaviour,

$$\widetilde{R} = \widetilde{R}_{\lim} + c_1|\sigma|^{-\lambda_1}, \quad \widetilde{U} = c_2|\sigma|^{-\lambda_2}, \quad \lambda_i > 0.$$

On the contrary, $a^{+\infty}$ is a saddle point. Such structure is the background for a proof of the existence of a trajectory which goes from $a^{+\infty}$ to $a^{-\infty}$ when τ varies from $+\infty$ to $-\infty$. This completes the proof of the main result

Theorem 1. *The solution of the problem (7), (8) is unique.*

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