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DEFINITIONS OF INFINITELY NARROW δ-SOLITONS IN A WEAK SENSE *

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Abstract

We consider different definitions in a weak sense of soliton-type solutions to KdV equation with small dispersion.

1. It is well known that the Korteweg-de Vries (KdV) equation

$$L_{KdV}[u] = u_t + (u^2)_x + \varepsilon^2 u_{xxx} = 0 \tag{1}$$

has the one-soliton solution

$$u(x,t,\varepsilon) = \frac{3v}{2}\cosh^{-2}(\frac{\sqrt{v}}{2}(x-vt)/\varepsilon), \quad x \in \mathbb{R}, \tag{2}$$

where v is the soliton velocity. The pointwise limit as $\varepsilon \to +0$ of solution (2) to the KdV equation is the discontinuous function $\frac{3v}{2}\chi(x-vt)$, where $\chi(\xi) = 1$ if $\xi = 0$ and $\chi(\xi) = 0$ if $\xi \neq 0$. The weak asymptotics (2) as $\varepsilon \to +0$, up to $O_{\mathcal{D}'}(\varepsilon^2)$, becomes the *infinitely narrow δ-soliton*

$$u_\varepsilon(x,t) = A\varepsilon\delta(x-vt), \quad \varepsilon \to +0, \qquad A = \frac{3v}{2}\int \cosh^{-2}(\frac{\sqrt{v}}{2}\xi)\, d\xi = 6\sqrt{v}, \tag{3}$$

and $\delta(x)$ is the Dirac delta function. Here and in what follows $\int$ denotes an improper integral from $-\infty$ to $+\infty$. By $O_{\mathcal{D}'}(\varepsilon^\alpha)$ we denote a distribution from $\mathcal{D}'(\mathbb{R})$ such that for any test function $\varphi(x) \in \mathcal{D}$ $\langle O_{\mathcal{D}'}(\varepsilon^\alpha), \varphi(x)\rangle = O(\varepsilon^\alpha)$, and $O(\varepsilon^\alpha)$ is understood in the ordinary sense.

We stress once more that here all generalized functions (distributions) are treated as functionals on the space $\mathcal{D}(\mathbb{R}_x)$ and these functionals depend on the other variables as on parameters.

*This research was supported by CONACYT under grant 41421 F.

It follows from (2) [2] that we have $u(x,t,\varepsilon) = O_{\mathcal{D}'}(\varepsilon)$ as $\varepsilon \to +0$ and $\varepsilon^2 u_{xxx} = O_{\mathcal{D}'}(\varepsilon^3)$. Therefore, the limit expression (3) was interpreted by V. P. Maslov, V. A. Tsupin and V. P. Maslov, G. A. Omel'yanov (see [1] and the references therein) as an asymptotic up to $O_{\mathcal{D}'}(\varepsilon^2)$ *generalized solution* of the Hopf equation

$$L_H[u] = u_t + (u^2)_x = 0, \tag{4}$$

which is the *limit problem* for the KdV equation. In the same works the corresponding generalized Hugoniot conditions, of the type of those for the shock wave front, were obtained.

If, instead of expression (2) which is the exact solution of the KdV equation and approximates the weak asymptotics (3), we consider the function

$$\tilde{u}(x,t,\varepsilon) = A\omega\Big(\frac{x-vt}{\varepsilon}\Big), \tag{5}$$

where $\omega(z) \in C^\infty(\mathbb{R})$ has a compact support or rapidly decreases as $|z| \to \infty$, $\int \omega(z)\,dz = 1$, then:

1) expression (5) also has asymptotics (3) in the sense of $\mathcal{D}'$ as $\varepsilon \to +0$ (see [2] for details);

2) substituting (5) into the Hopf equation (4), provided that a certain correlation between the constants v and A is true (the generalized Hugoniot condition), we have $L_H[\tilde{u}] = O_{\mathcal{D}'}(\varepsilon^2)$;

3) and $u(x,t,\varepsilon) - \tilde{u}(x,t,\varepsilon) = O_{\mathcal{D}'}(\varepsilon^2)$.

Therefore, an asymptotics up to $O_{\mathcal{D}'}(\varepsilon^2)$, i.e., an infinitely narrow δ-soliton-type solution of the KdV equation (or the Hopf equation which is the limit problem of the KdV equation), can be sought starting not from the exact solution (2) of the KdV equation (which is a regularization of the Hopf equation) but from an ansatz of the form (5) substituted *directly* into the Hopf equation.

Generalizing (3) and (5), one can seek the solution in the form

$$u(x,t,\varepsilon) = u_0(x,t) + g(t)\omega\Big(\frac{x-\phi(t)}{\varepsilon}\Big),$$

by substituting this singular ansatz into the Hopf equation.

However, such an ansatz results in the solution with constant amplitude $g = \text{const}$ of the soliton for $u_0 \neq \text{const}$, which contradicts the well-known results about the soliton behavior. Therefore, generalizing formula (5), we can attempt to construct an asymptotic solution to the KdV equation (or the Hopf equation) of the form

$$u^*(x,t,\varepsilon) = u_0(x,t) + g(t)\varepsilon\delta(x-\phi(t),\varepsilon) + e(x,t)\varepsilon\theta(x-\phi(t),\varepsilon), \quad \varepsilon > 0, \tag{6}$$

where $u_0(x,t)$, $g(t)$, $e(x,t)$, $\phi(t)$ are the desired smooth functions, and $\varepsilon\delta(x,\varepsilon) = \omega(\frac{x}{\varepsilon})$, $\varepsilon\theta(x,\varepsilon) = \varepsilon\omega_0\big(\frac{x}{\varepsilon}\big)$ are smooth approximations of the distributions $\varepsilon\delta(x)$ and $\varepsilon\theta(x)$, respectively.

Here, the function $\omega(z) \in C^\infty(\mathbb{R})$ either has a compact support or decreases sufficiently rapidly as $|z| \to \infty$, for example, $|\omega(z)| \le C(1+|z|)^{-3}$ and $\int \omega(z)\,dz = 1$; $\omega_0(z) \in C^\infty(\mathbb{R})$, $\lim_{z\to+\infty}\omega_0(z) = 1$, $\lim_{z\to-\infty}\omega_0(z) = 0$.

Then we have in the sense of $\mathcal{D}'(\mathbb{R})$ (see the notation above) $\varepsilon\delta(x,\varepsilon) = \varepsilon\delta(x) + O_{\mathcal{D}'}(\varepsilon^2)$, $\varepsilon\theta(x,\varepsilon) = \varepsilon\theta(x) + O_{\mathcal{D}'}(\varepsilon^2)$, $\varepsilon \to +0$. For more details, see [2].

Now, following [2], we can introduce the definition of the asymptotic generalized solution of the form (6). Namely, we call $u^*(x,t,\varepsilon)$ (8) a weak *asymptotic solution* to the KdV equation (1) if

$$L_{KdV}[u^*(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon^2). \tag{7}$$

It is easy to see that our definition of the solution can *depend* on the choice of approximations $\frac{1}{\varepsilon}\omega(\frac{x-\phi(t)}{\varepsilon})$ and $\omega_0(\frac{x-\phi(t)}{\varepsilon})$ to the distributions $\delta(x-\phi(t))$, and $\theta(x-\phi(t))$, respectively. Actually, the dynamics of solution of the type (6) is *independent* of the approximation of the Heaviside function $\omega_0(\frac{x-\phi(t)}{\varepsilon})$ [2].

In order to obtain the results known from the KdV equation theory it seems natural to use the function from the formula for the exact one-soliton solution (2) to the KdV equation as an approximation for $\varepsilon\delta(x-\phi(t),\varepsilon)$.

The system for the functions $u_0(x,t)$, $g(t)$, $e(x,t)$, $\phi(t)$ follows from definitions (7) (this system was derived in detail in [2])

$$\begin{aligned}
u_{0t} + (u_0^2)_x &= 0,\\
\phi_t - 2u_0(\phi(t),t) - \frac{2}{3}g(t) &= 0,\\
e(\phi(t),t) - \frac{3\sqrt{6}}{2}g_t(t)/g^{3/2}(t) &= 0,\\
\big(e_t(x,t) + 2(u_0(x,t)e(x,t))_x\big)\Big|_{x>\phi(t)} &= 0.
\end{aligned} \tag{8}$$

It is easy to verify that under the condition $g > 0$ (which is an analog of the admissibility condition in the theory of shock waves) the solution of system (8) exists on any interval $t \in [0,\ T]$ such that the smooth solution u_0 of the Hopf equations exists on this interval.

System (8) can be solved in the following way: first, one finds the smooth solution of the Hopf equation, next, one finds the function $e(x,t)$ from the last equations (which is uniquely solvable in view of the inequality $2u_0(\phi,t) < \phi_t$), then one finds the (positive) function $g(t)$ from the next to the last equation, and finally, one finds the function $\phi(t)$.

Note that system (8) contains no obstacles to setting $e(x,t) = 0$. If so, $g(t) = \text{const}$ in the case of an arbitrary (nonconstant) background function $u_0(x,t)$. But this conclusion is contrary to well known properties of soliton solutions of the KdV equation (see, e.g., [1]).

Moreover, under our notation, the weak asymptotics of the asymptotic one-soliton solution to the KdV equation, constructed by V. P. Maslov and G. A. Omel'yanov [1], has the form

$$u^*_{1,\varepsilon}(x,t) = u_{01}(x,t) + g_1(t)\varepsilon\delta(x-\phi_1(t)) + e_1(x,t)\varepsilon[1-\theta(x-\phi_1(t))], \quad \varepsilon \to +0. \tag{9}$$

In other words, in the case (6) the "shock wave" with a small amplitude $\varepsilon e(x,t)\theta(x-\phi_1(t))$ propagates *in front of the soliton* $\varepsilon\delta(x-\phi_1(t))$, but in the asymptotic one-soliton solution constructed in [1] the small shock wave $\varepsilon e_1(x,t)[1-\theta(x-\phi_1(t))]$ arises *behind the soliton* .

If we apply definition (7) to the asymptotic solution obtained in [1], whose weak asymptotics yields (9), we obtain the following system of equations [2]:

$$
\begin{aligned}
u_{01t} + (u_{01}^2)_x &= 0,\\
\phi_{1t} - 2u_{01}(\phi_1(t),t) - \frac{2}{3}g_1(t) &= 0,\\
e_1(\phi(t),t) + \frac{3\sqrt{6}}{2}g_{1t}(t)/g_1^{3/2}(t) &= 0,\\
\big(e_{1t}(x,t) + 2(u_{01}(x,t)e_1(x,t))_x\big)\Big|_{x<\phi_1(t)} &= 0.
\end{aligned}
\tag{10}
$$

The solution of the last system for $g_{1t}(t) \neq 0$ is not uniquely determined by the initial conditions $e_1(x,0)$ for $x \leq \phi_1(0)$, since the velocity along the characteristic ($\dot{x} = 2u_{01}(x(t),t)$) is less (for $g_1(t) > 0$) than the velocity of the soliton $\phi_{1t} = 2u_{01}(\phi_1(t),t) + \frac{2}{3}g_1(t)$ given by (10).

Thus, the assumption that the structure of the solution to the KdV equation is specified by (9) due to definition (7) leads to an ill-posed Cauchy problem (with a nonunique solution) for the functions $u_{01}(x,t),\ \ g_1(t),\ \ e_1(x,t),\ \ \phi_1(t)$.

On the other hand, the system of equations obtained in [1] for these functions has the form

$$
\begin{aligned}
u_{01t} + (u_{01}^2)_x &= 0,\\
\phi_{1t} - 2u_{01}(\phi_1(t),t) - \frac{2}{3}g_1(t) &= 0,\\
e_1(\phi(t),t) + \frac{3\sqrt{6}}{2}g_{1t}(t)/g_1^{3/2}(t) &= 0,\\
\big(e_{1t}(x,t) + 2(u_{01}(x,t)e_1(x,t))_x\big)\Big|_{x<\phi_1(t)} &= 0,\\
g_1(t) + 2u_{01}(\phi_1(t),t) &= \text{const},
\end{aligned}
\tag{11}
$$

It is evident that this system differs from system (10) by the additional equation $g_1(t) + 2u_{01}(\phi_1(t),t) = g_1(0) + 2u_{01}(\phi_1(0),0)$. The presence of this equation implies that system (11) splits into the two systems: (the first, second, and last equations; the third and fourth equations from (11)).

In this case, the third equality in (11) is the boundary condition for the fourth equation in (11), which turns the Cauchy problem for the fourth equation in (11) into the well-posed one (the Cauchy condition, in view of (9), has the form $e_1(x,0) = e_1^0(x)[1 - \theta(x - \phi_1(0))]$).

One can show [2] that the weak asymptotics corresponding to the asymptotic solution of the Cauchy problem for the KdV equation constructed in [1] cannot be derived from the solution to the KdV equation with the help of definition (7), and vice versa.

Definition 1. *[2] The function $u^*(x,t,\varepsilon) = u_0(x,t) + g(t)\varepsilon\delta(x - \phi(t)) + e(x,t)\varepsilon\theta(-x + \phi(t))$ is a weak asymptotic (soliton-type) solution to the KdV equation (1) for $t \in [0,T]$ if for any constants $c_1,\ \ c_2$ the following equality holds*

$$
\begin{aligned}
\big(c_1 + c_2u^*(x,t,\varepsilon)\big)L_{KdV}[u^*(x,t,\varepsilon)] &= O_{\mathcal{D}'}(\varepsilon^2),\\
u_\varepsilon^*(x,0) &= u_\varepsilon^{0*}(x) + O_{\mathcal{D}'}(\varepsilon^2),
\end{aligned}
\tag{12}
$$

It is clear that (12) is equivalent to the following relations

$$L_{KdV}[u^*(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon^2), \qquad u^*(x,t,\varepsilon)L_{KdV}[u^*(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon^2),$$

and that the first relation coincides with definition (7).

One can easily see that (12) can be rewritten as an integral identity but of an unusual form.

By analogy to what was previously said, the solution depends on the choice of the approximation, and to obtain the results known in the theory of the KdV equation one should choose, as an approximation of the asymptotic distribution $g(t)\varepsilon\delta(x-\phi(t))$, the function from the formula for asymptotic solution to the KdV equation [1, 2]:

$$g(t)\varepsilon\delta(x-\phi(t),\varepsilon) = g(t)\omega\big(\alpha(t)\frac{x-\phi(t)}{\varepsilon}\big),$$

where $\alpha(t) = \sqrt{\frac{g(t)}{6}}$, $\omega(z) = \cosh^{-2}(z)$.

It is well known that the function $\omega(\alpha(t)\tau)$ is a solution of the boundary value problem for the differential equation

$$-\phi_t(t)\frac{d\omega}{dz} + 2(u_0(\phi(t),t) + g(t)\omega)\frac{d\omega}{dz} + \alpha^2(t)\frac{d^3\omega}{dz^3} = 0, \tag{13}$$

where $\omega(z) \to 0$ as $|z| \to \infty$.

As shown in [2], Definition 1 implies the system derived and justified by V. Maslov and G. Omel'yanov (see the references in [1]).

However, as already mentioned, this is obtained under an additional assumption on the form of the approximation of the δ-function. Namely, this approximation must be determined by the solution of Eq. (13) which does *not* follow from Definition 1.

It turns out that this obstacle is removed by the following natural definition.

Definition 2. *The function $u_N^*(x,t,\varepsilon)$ is a weak asymptotic solution of order infinity if for any $N>0$ the following relation holds:*

$$L_{KdV}[u_N^*(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon^N).$$

Here we consider a special case of this situation and prove the following statement.

Teorem 1. *Let $u_N^*(x,t,\varepsilon) = A\omega(\frac{x-Vt}{\varepsilon})$ be a weak asymptotic solution of the KdV equation in the sense of Definition 2. Then the function ω is a solution of the equation*

$$A\omega^2 - V\omega + \omega'' = 0,$$

which belongs to the Schwartz space S.

Proof. Suppose that $\omega \in S(\mathbf{R})$ and A, ν are constants. We seek the solution of the KdV equation in the form

$$u(x,t,\varepsilon) = A\omega(\frac{x-\nu t}{\varepsilon}) \tag{14}$$

with accuracy $O_{\mathcal{D}'}(\varepsilon^n)$, where n is arbitrary. Using the definition of the weak asymptotic solution, we obtain the system of equations

$$\begin{aligned} A\overline{\Omega}_0 - \nu\Omega_0 &= 0, \\ \nu\Omega_1 - A\overline{\Omega}_1 &= 0, \\ (k+1)k\Omega_{k-2} - \nu\Omega_k + A\overline{\Omega}_k &= 0, \qquad k \geq 2. \end{aligned} \tag{15}$$

Here $\Omega_k = \int \omega(z)z^k\,dz$ and $\overline{\Omega}_k = \int \omega(z)^2 z^k\,dz$. We denote $\varphi(z) = A\omega(z)^2 - \nu\omega(z) + \omega''(z)$. Then system (21) implies that $\int \varphi(z)z^k\,dz = 0$ for all $k \in \mathbf{Z}_+$. Now we need the following assertion.

Lemma 1. *Let $\varphi \in S(\mathbf{R})$, and let $\int \varphi(z)z^k\,dz = 0$ for all $k \in \mathbf{Z}_+$. Then $\varphi(z) \equiv 0$.*

Proof. Let $\tilde{\varphi}$ be the Fourier transform of φ. The assumption of the lemma is equivalent to $\tilde{\varphi}^{(k)}(0) = 0$ for all $k \geq 0$. However, this does not allow us to conclude that $\varphi(x) \equiv 0$ because the function φ is not analytic.

We consider the functions

$$f_k(\xi) = \int_{-\infty}^{\infty} \varphi(x)x^k e^{ix\xi - x^2/2}\,dx.$$

It is easy to see that for any $k \in \mathbf{Z}$ we have

$$f_k(0) = \int_{-\infty}^{\infty} e^{-x^2/2}\varphi(x)x^k\,dx = 0.$$

Let us consider the function

$$w(z) = \int_{-\infty}^{\infty} \varphi(x)e^{ixz - x^2/2}\,dx \tag{16}$$

of the complex variable z. Since the integral (16) converges and the integrals obtained from (16) by the formal (k-multiple, where k is arbitrary) differentiation with respect to z under the sign of integral converge uniformly on any compact set in $\mathbf{C}$, we see that $w(z)$ is an entire analytic function such that

$$w^{(k)}(0) = i^k \int_{-\infty}^{\infty} \varphi(x)x^k e^{-x^2/2}\,dx = i^k f_k(0) = 0.$$

This means that $w(z) \equiv 0$ and thus we have

$$e^{-x^2/2}\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w(z)e^{-ixz}\,dz = 0,$$

i.e., $\varphi(x) \equiv 0$.

Thus we have the following differential equation for ω:

$$A\omega^2 - \nu\omega + \omega'' = 0$$

and the condition $\omega \in \mathbf{S}$. The solution of this problem is the function

$$\omega(z) = \frac{3\nu}{2A}\operatorname{sech}^2\Big(c + \frac{\sqrt{\nu}z}{2}\Big).$$

References

[1] Maslov, V.P.; Omel'yanov, G.A. (2001) Geometric Asymptotics for Nonlinear PDE. I. *AMS Transl. Vol. 202.*

[2] Danilov, V.G.; Shelkovich, V.M. (2002) Propagation of infinitely narrow δ-solitons. http://arXiv.org/abs/math-ph/0012002.